ON THE REFLECTION OF SOUND WAVES FROM A PLANE WITH A MOVABLE PART IN THE FORM OF A CYLINDRICAL PISTON

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PMM Vol.24, No.4, 1960, pp. 726-731

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(Received 21 April 1960)

This paper is concerned with the problem of the reflection of a plane sound wave from a rigid plane which has a movable part in the form of a rigid cylindrical piston. The force acting upon the piston from the side of the fluid is determined. An integro-differential equation of motion of the piston is constructed and its solution is given.

1. Assume that the plane sound wave, which has a pressure profile

$$p = p_0\left(t + \frac{z}{c}\right), \qquad p_0(t) = 0 \qquad \text{at} \quad t \leqslant 0$$

encounters the plane $z = 0$ at the instant $t = 0$ and is reflected from it. After the reflection, in the axisymmetric case, the deformable part of the plane will move with a velocity $V_z = V(r, t)$, where $V(r, 0) = 0$. We assume that the deformations are small, and we determine the pressure for $t > 0$. In order to do this, it is necessary to solve the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p \qquad (1.1)$$

subject to the conditions

$$\frac{\partial p}{\partial z} = -\rho_0 \frac{\partial V}{\partial t} \qquad \text{at} \quad z = 0$$

$$p = p_0(2t) \qquad \text{at} \quad z = ct$$

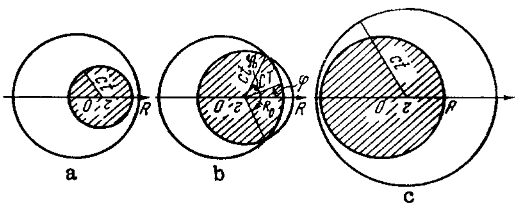

Fig. 1.

Here c = const is the speed of sound in the fluid, ρ_0 = const is the density of the fluid. Let

$$p = p_1(t, z) + p_2(r, t, z)$$

where $p_1(t, z)$ is the solution corresponding to the reflection from a rigid plane [1]

$$p_1(t, z) = p_0\left(t + \frac{z}{c}\right) + p_0\left(t - \frac{z}{c}\right)$$

Then, in order to determine $p_2(r, t, z)$ it is necessary to solve Equation (1.1) with the conditions

$$\frac{\partial p_2}{\partial z} = -\rho_0 \frac{\partial V}{\partial t} \quad \text{at} \quad z = 0, \qquad p_2 = 0 \quad \text{at} \quad z = ct \qquad (1.2)$$

(the first condition holds at the deformable part of the plane). The solution of Equation (1.1) with the conditions (1.2) is given in the form [2]

$$p_2(r, t, z) = \frac{c\rho_0}{2\pi}\int_0^{2\pi} d\varphi \int_{z/c}^{t} V_t'(\eta, t - \tau)\, d\tau \qquad \left(\begin{array}{l}\eta = \sqrt{r^2 + L^2 + 2rL\cos\varphi} \\ L = \sqrt{c^2\tau^2 - z^2}\end{array}\right)$$

2. Let the deforming part of the plane be represented by a movable rigid piston placed into a cutout in the plane. In this case $V = V(t)$ and at the surface of the piston $z = 0$

$$p_2(r, t, 0) = \frac{c\rho_0}{2\pi}\iint_S V'(t - \tau)\, d\varphi\, d\tau$$

With fixed r and t the limits of integration with respect to ϕ and τ are determined from the conditions of the intersection of a circle of radius R (R is the radius of the piston) with a circle of radius ct whose center lies at a distance r from the axis of the piston (Fig. 1).

Depending on the values of r and t three cases are possible:

First Case. $0 \leqslant t < (R - r)/c$; (Fig. 1,a)

$$p_2(r, t, 0) = \frac{c\rho_0}{2\pi}\int_0^{2\pi} d\varphi \int_0^{t} V'(t - \tau)\, d\tau = c\rho_0 V(t)$$

The total pressure is

$$p(r, t, 0) = 2p_0(t) + c\rho_0 V(t) \qquad (2.1)$$

Second Case. $(R - r)/c \leqslant t < (R + r)/c$; (Fig. 1,b)

$$p_2(r,t,0)=\frac{c\rho_0}{2\pi}\Big[\int\limits_{\varphi_0}^{2\pi-\varphi_0}d\varphi\int\limits_0^t V'(t-\tau)\,d\tau+\int\limits_{-\varphi_0}^{\varphi_0}d\varphi\int\limits_0^T V'(t-\tau)\,d\tau\Big]=$$

$$=c\rho_0V(t)-\frac{c\rho_0}{\pi}\int\limits_0^{\varphi_0}V(t-T)\,d\varphi$$

The total pressure is

$$p(r,t,0)=2p_0(t)+c\rho_0V(t)-\frac{c\rho_0}{\pi}\int\limits_0^{\varphi_0}V(t-T)\,d\varphi \tag{2.2}$$

Third Case. $(R+r)/c<t<\infty$ (Fig. 1,c)

$$p_2(r,t,0)=\frac{c\rho_0}{2\pi}\int\limits_0^{2\pi}d\varphi\int\limits_0^T V'(t-\tau)\,d\tau=c\rho_0V(t)-\frac{c\rho_0}{\pi}\int\limits_0^{\pi}V(t-T)\,d\varphi.$$

The total pressure is

$$p(r,t,0)=2p_0(t)+c\rho_0V(t)-\frac{c\rho_0}{\pi}\int\limits_0^{\pi}V(t-T)\,d\varphi \tag{2.3}$$

Here

$$\varphi_0=\cos^{-1}\frac{R^2-r^2-c^2t^2}{2rct},\qquad T=\sqrt{R^2-r^2\sin^2\varphi}-r\cos\varphi,\qquad T(r,\varphi_0)=t$$

The compressive force, which acts upon the piston from the fluid side, is

$$F(t)=\iint\limits_{r\leqslant R}p(r,t,0)\,d\sigma=2\pi\int\limits_0^R p(r,t,0)\,rdr$$

At $t=0$ a cylindrical wave appears near the edges of the piston, which then propagates with velocity c towards the center of the piston. The wave front $r^\circ=R-ct$ divides the surface of the piston into two regions. For $t<R/c$ in the first region ahead of the wave front, where $0\leqslant r<r^\circ$, $t<(R-r)/c$, the pressure is determined from the formula (2.1); in the second region behind the wave front, where $r^\circ<r<R$, $(R-r)/c<t\leqslant(R+r)/c$, the pressure is determined from the formula (2.2). Thus we shall have

$$F(t)=2\pi\int\limits_0^{R-ct}[2p_0(t)+c\rho_0V(t)]\,rdr+2\pi\int\limits_{R-ct}^{R}\Big\{2p_0(t)+c\rho_0V(t)-$$

$$-\frac{c\rho_0}{\pi}\int\limits_0^{\varphi_0(r,t)}V[t-T(r,\varphi)]\,d\varphi\Big\}\,rdr=\pi R^2[2p_0(t)+c\rho_0V(t)]-2c\rho_0\Phi_1$$

$$\Phi_1 = \int_{R-ct}^{R} rdr \int_0^{\varphi_0(r,t)} V\,[t - T\,(r, \varphi)]\,d\varphi$$

In order to evaluate the integral Φ_1 we let

$$t - T\,(r, \varphi) = \tau, \qquad \tau\,(r, 0) = t - (R - r)\,/\,c, \qquad \tau\,(r, \varphi_0) = 0$$

and, after interchanging the order of integration, we obtain

$$\Phi_1 = \int_0^t V\,(\tau)\,d\tau \int_{R-ct}^{R} \frac{\Psi\,(r)\,rdr}{(t-\tau)\sqrt{1-\Psi^2(r)}} = cR\int_0^t \sqrt{1-\left[\frac{c\,(t-\tau)}{2R}\right]^2}\,V\,(\tau)\,d\tau$$

$$\Psi\,(r) = \frac{R^2 - r^2 + c^2\,(t-\tau)^2}{2Rc\,(t-\tau)} \tag{2.4}$$

At the instant $t = R/c$ the wave reflects at the center and then goes to the edge; the wave front is at $r^\circ = ct - R$. In the time interval $R/c \leqslant t < 2R/c$ in the first region ahead of the reflected wave front, where $r^\circ < r < R$, $R/c \leqslant t < (R + r)/c$, the pressure is determined by Formula (2.3). In the second region behind the reflected wave front, where $0 < r \leqslant r^\circ$, $(R + r)/c < t \leqslant 2R/c$, the pressure is determined by Formula (2.2). As a result we obtain

$$F\,(t) = 2\pi \int_{ct-R}^{R} \left\{2p_0\,(t) + c\rho_0 V\,(t) - \frac{c\rho_0}{\pi}\int_0^{\pi} V\,[t - T\,(r, \varphi)]\,d\varphi\right\} rdr +$$

$$+ 2\pi \int_0^{ct-R} \left\{2p_0\,(t) + c\rho_0 V\,(t) - \frac{c\rho_0}{\pi}\int_0^{\varphi_0(r,t)} V\,[t - T\,(r, \varphi)]\,d\varphi\right\} rdr$$

$$= \pi R^2\,[2p_0\,(t) + c\rho_0 V(t)] - 2c\rho_0 \Phi_2$$

Here

$$\Phi_2 = \int_0^{ct-R} rdr \int_0^{\varphi_0(r,t)} V\,[t - T\,(r, \varphi)]\,d\varphi + \int_{ct-R}^{R} rdr \int_0^{\pi} V\,[t - T\,(r, \varphi)]\,d\varphi = \Phi_1$$

This result is obtained by calculations similar to those performed before.

For the time interval $2R/c < t < \infty$ the pressure is determined by Formula (2.3); thus

$$F\,(t) = 2\pi \int_0^R \left\{2p_0\,(t) + c\rho_0 V\,(t) - \frac{c\rho_0}{\pi}\int_0^{\pi} V\,[t - T\,(r, \varphi)]\,d\varphi\right\} rdr$$

$$= \pi R^2\,[2p_0\,(t) + c\rho_0 V\,(t)] - 2c\rho_0 \Phi_3$$

After some calculations we have

$$\Phi_3 = \int_0^R r dr \int_0^\pi V\,[t - T\,(r,\ \varphi)]\, d\varphi = cR \int_{t-2R/c}^{t} \sqrt{1 - \left[\frac{c\,(t-\tau)}{2R}\right]^2} V\,(\tau)\, d\tau$$

Thus, for the time interval $0 \leqslant t \leqslant 2R/c$, in which the wave that comes from the edge of the piston will reach the center and after reflection at the center will return to the edge, we have

$$F\,(t) = \pi R^2\,[2p_0\,(t) + c\rho_0 V\,(t)] - 2c^2\rho_0 R \int_0^t \sqrt{1 - \left[\frac{c\,(t-\tau)}{2R}\right]^2} V\,(\tau)\, d\tau \qquad (2.5)$$

For the time interval $2R/c < t \leqslant \infty$ we have

$$F\,(t) = \pi R^2\,[2p_0\,(t) + c\rho_0 V\,(t)] - 2c^2\rho_0 R \int_{t-2R/c}^{t} \sqrt{1 - \left[\frac{c\,(t-\tau)}{2R}\right]^2} V\,(\tau)\, d\tau \qquad (2.6)$$

3. Let us construct the equation of motion of the piston. We denote its displacement by $u(t)$, and obtain

$$\pi R^2 h\rho \frac{d^2u}{dt^2} = F\,(t) - F_r \qquad \left(\frac{du}{dt} = -V\,(t)\right)$$

Here ρ is the density, h is the thickness of the piston, F_r is the reaction force which acts upon the piston from the opposite side. Assume that

$$F_r = c_1 \frac{du}{dt} + c_2 u$$

Now we shall transform to nondimensional quantities

$$t_1 = \frac{ct}{2R}, \qquad u_1 = \frac{u}{2R}, \qquad p_{01} = 4\frac{Rp_0}{h\rho c^2}$$

After dropping the subscript 1 we obtain the equation of motion of the piston in the form

$$u''\,(t) + 2\alpha u'\,(t) + \beta u\,(t) = p\,(t) + \varepsilon \int_0^t \sqrt{1-(t-\tau)^2}\, u'\,(\tau)\, d\tau \qquad (0 \leqslant t \leqslant 1) \qquad (3.1)$$

$$u''\,(t) + 2\alpha u'\,(t) + \beta u\,(t) = p\,(t) + \varepsilon \int_{t-1}^{t} \sqrt{1-(t-\tau)^2}\, u'\,(\tau)\, d\tau \qquad (1 \leqslant t < \infty) \qquad (3.2)$$

where

$$\alpha = \frac{\pi}{8}\varepsilon + \frac{c_1}{Rh\rho c}, \qquad \beta = \frac{4c_2}{\pi h\rho c^2}, \qquad \varepsilon = \frac{8}{\pi}\frac{R\rho_0}{h\rho} \qquad (3.3)$$

The initial conditions have the form

$$u(0) = 0, \qquad u'(0) = 0 \tag{3.4}$$

From this we see that the motion of a cylindrical piston under the action of an incident wave is described by an integro-differential equation, which cannot be reduced to a differential equation, unlike that of the slab-like piston [1]. Note that for $1 < t < 2$

$$\int_{t-1}^{t} \sqrt{1-(t-\tau)^2}\,u'(\tau)\,d\tau = \int_{1}^{t} \sqrt{1-(t-\tau)^2}\,u'(\tau)\,d\tau + \int_{t-1}^{1} \sqrt{1-(t-\tau)^2}\,u'(\tau)\,d\tau$$

Here the second integral on the right-hand side is known if we find the solution for $t < 1$, and in general for $n < t < n+1$ $(n = 1, 2, \ldots)$

$$\int_{t-1}^{t} \sqrt{1-(t-\tau)^2}\,u'(\tau)\,d\tau = \int_{n}^{t} \sqrt{1-(t-\tau)^2}\,u'(\tau)\,d\tau + \int_{t-1}^{n} \sqrt{1-(t-\tau)^2}\,u'(\tau)\,d\tau$$

where the second integral on the right-hand side is known if we find the solution for $t < n$. The solution of Equations (3.1) and (3.2) can be represented in the form

$$u(t) = \int_0^t p(t-\tau)\,q(\tau)\,d\tau \qquad (0 \leqslant t \leqslant 1) \tag{3.5}$$

$$u(t) = \int_0^{t-n} p(t-\tau)\,q(\tau)\,d\tau + \varepsilon \int_0^{t-n} \Big[\int_{t-\tau-1}^{n} \sqrt{1-(t-\tau-\xi)^2}\,u'(\xi)\,d\xi -$$

$$- u(n)\sqrt{1-(t-\tau-n)^2} \Big] q(\tau)\,d\tau + u(n)\,q'(t-n) + [u'(n) + 2\alpha u(n)]\,q(t-n)$$

$$(n \leqslant t \leqslant n+1) \tag{3.6}$$

where $q(t)$ is the solution of the equation

$$q''(t) + 2\alpha q'(t) + \beta q(t) - \varepsilon \int_0^t \sqrt{1-(t-\tau)^2}\,q'(\tau)\,d\tau = 0 \qquad (0 \leqslant t \leqslant 1) \tag{3.7}$$

with the initial conditions

$$q(0) = 0, \qquad q'(0) = 1 \tag{3.8}$$

4. Let us find the exact solution of Equation (3.7) with the conditions (3.8). If it is assumed that

$$\int_0^t \sqrt{1-(t-\tau)^2}\,q'(\tau)\,d\tau = q(t) - \int_0^t \frac{t-\tau}{\sqrt{1-(t-\tau)^2}}\,q(\tau)\,d\tau$$

then Equation (3.7) can be written as follows:

$$q''(t) + 2\alpha q'(t) + (\beta - \varepsilon)\,q(t) = -\,\varepsilon \int_0^t \frac{t-\tau}{\sqrt{1-(t-\tau)^2}}\,q(\tau)\,d\tau \tag{4.1}$$

We shall seek a solution in the form of a series in the parameter ϵ, which is determined according to (3.3), by letting

$$q(t) = q_0(t) + \varepsilon q_1(t) + \varepsilon^2 q_2(t) + \ldots \tag{4.2}$$

Substituting the series (4.2) into Equation (4.1), collecting terms which contain equal powers of ϵ, and equating their sum to zero leads to a system of ordinary differential equations with constant coefficients:

$$q_0''(t) + 2\alpha q_0'(t) + (\beta - \varepsilon) q_0(t) = 0 \tag{4.3}$$

$$q_n''(t) + 2\alpha q_n'(t) + (\beta - \varepsilon) q_n(t) = -\int_0^t \frac{t-\tau}{\sqrt{1-(t-\tau)^2}} q_{n-1}(\tau)\, d\tau \qquad (n = 1, 2, \ldots) \tag{4.4}$$

with the initial conditions

$$q_n(0) = 0 \quad (n = 0, 1, \ldots), \qquad q_0'(0) = 1, \qquad q_n'(0) = 0 \quad (n = 1, 2, \ldots) \tag{4.5}$$

These equations can be easily solved successively:

$$q_0(t) = \frac{1}{2\omega}(e^{\lambda_1 t} - e^{\lambda_2 t}), \qquad \lambda_{1,2} = -\alpha \pm \omega, \qquad \omega = \sqrt{\alpha^2 + \varepsilon - \beta} \tag{4.6}$$

where ω can also be imaginary

$$q_n(t) = -\int_0^t q_0(t-\tau)\, d\tau \int_0^\tau \frac{\tau-\xi}{\sqrt{1-(\tau-\xi)^2}} q_{n-1}(\xi)\, d\xi$$

$$= -\int_0^t q_{n-1}(\xi)\, d\xi \int_\xi^t \frac{\tau-\xi}{\sqrt{1-(\tau-\xi)^2}} q_0(t-\tau)\, d\tau$$

$$= -\int_0^t q_{n-1}(\xi)\, d\xi \int_0^{t-\xi} \frac{t-\xi-\tau}{\sqrt{1-(t-\xi-\tau)^2}} q_0(\tau)\, d\tau$$

or

$$q_n(t) = -\int_0^t K(t-\tau)\, q_{n-1}(\tau)\, d\tau, \qquad K(t) = \int_0^t \frac{t-\tau}{\sqrt{1-(t-\tau)^2}} q_0(\tau)\, d\tau \tag{4.7}$$

We shall prove the convergence of the series (4.2) by utilizing obvious estimates

$$|q_0(t)| < M, \qquad M = \max|q_0(t)|, \qquad |K(t)| \leqslant \int_0^t \frac{t-\tau}{\sqrt{1-(t-\tau)^2}} |q_0(\tau)|\, d\tau < M$$

$$|q_1(t)| \leqslant \int_0^t |K(t-\tau)| \cdot |q_0(\tau)|\, d\tau < M^2 t, \qquad |q_2(t)| < M^3 \frac{t^2}{2!}$$

$$|q_n(t)| < M^{n+1} \frac{t^n}{n!}$$

Then

$$\left|\sum_{n=0}^{\infty} \varepsilon^n q_n(t)\right| \leqslant \sum_{n=0}^{\infty} \varepsilon^n \, |q_n(t)| < \sum_{n=0}^{\infty} \varepsilon^n M^{n+1} \frac{t^n}{n!} = Me^{\varepsilon Mt}$$

Thus, the exact solution of Equation (3.7) with the conditions (3.8) is represented in the form of a series (4.2) which converges uniformly for all values of $t < 1$ and any parameter ϵ.

However, for $\epsilon > 1$ the solution in the form of the series is inconvenient from practical considerations. We shall derive an approximate solution. For this purpose we shall expand the root $\sqrt{1 - t^2}$ in a series

$$\sqrt{1-t^2} = \sum_{n=0}^{\infty} a_n \cos \gamma_n t \qquad \left(\gamma_n = \frac{2n+1}{2}\pi\right) \tag{4.8}$$

$$\int_0^1 \sqrt{1-t^2} \cos \gamma_n t \, dt = \frac{\pi}{2} \frac{J_1(\gamma_n)}{\gamma_n} = \frac{a_n}{2}, \quad \text{or} \quad a_n = \pi \frac{J_1(\gamma_n)}{\gamma_n}$$

where J_1 is the Bessel function of the first order. One can show that the series (4.8) converges uniformly for all t and rather rapidly at that, since it follows from the properties of Bessel functions that

$$a_n \approx \frac{(-1)^n \sqrt{\pi}}{\gamma_n^{3/2}} \quad \text{for large } n$$

We write out several of the first terms of the series (4.8):

$$\sqrt{1-t^2} = 1.133 \cos \frac{\pi}{2} t - 0.188 \cos \frac{3\pi}{2} t + 0.084 \cos \frac{5\pi}{2} t - 0.050 \cos \frac{7\pi}{2} + \ldots$$

In the first approximation we retain only one term of the series, i.e. we let

$$\sqrt{1-t^2} \approx 1.133 \cos \frac{\pi}{2} t \tag{4.9}$$

Substitution of (4.9) into Equation (3.7) yields

$$L\{q\} = q''(t) + 2\alpha q'(t) + \beta q(t) - \varepsilon_1 \int_0^t \cos \frac{\pi}{2}(t-\tau)\, q'(\tau)\, d\tau = 0$$

$$q(0) = 0, \qquad q'(0) = 1 \qquad (\varepsilon_1 = 1.133\, \varepsilon)$$

Let us apply the one-sided Laplace transform and denote by $q^*(\lambda)$ the transformation of the function $q(t)$:

$$\int_0^{\infty} L\{q\} e^{-\lambda t} dt = \left(\lambda^2 + 2\alpha\lambda + \beta - \varepsilon \frac{\lambda^2}{\lambda^2 + {}^1/_4\, \pi^2}\right) q^*(\lambda) - 1 = 0$$

From this

$$q^*(\lambda) = \frac{\psi(\lambda)}{\varphi(\lambda)}, \qquad \psi(\lambda) = \lambda^2 + \frac{1}{4}\pi^2$$

$$\varphi(\lambda) = \lambda^4 + 2\alpha\lambda^3 + (\tfrac{1}{4}\pi^2 + \beta - \varepsilon)\lambda^2 + \\ + \tfrac{1}{2}\pi^2\alpha\lambda + \tfrac{1}{4}\pi^2\beta \qquad (4.10)$$

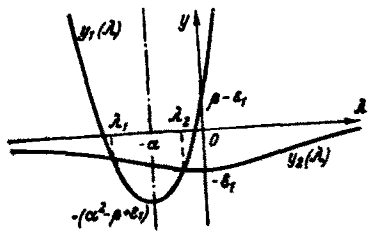

Fig. 2.

In order to find the inverse transform $q(t)$ it is necessary to know the roots of the function $\phi(\lambda)$. Let us study the roots of the equation $\phi(\lambda) = 0$ by replacing it by its equivalents

$$y_1(\lambda) = y_2(\lambda), \qquad y_1(\lambda) = \lambda^2 + 2\alpha\lambda + \beta - \varepsilon, \qquad y_2(\lambda) = -\frac{\pi^2}{4}\frac{\varepsilon_1}{\lambda^2 + {}^1/_4\pi^2}$$

The real roots of the equation are those values of λ for which the curves of the functions $y_1(\lambda)$ and $y_2(\lambda)$ intersect each other (Fig. 2).

Two cases are possible. If

$$2\alpha^2 \geqslant \beta - \varepsilon_1 + \sqrt{(\beta - \varepsilon_1)^2 + \tfrac{1}{2}\pi^2(\beta + \varepsilon_1) + \tfrac{1}{16}\pi^4} - \tfrac{1}{4}\pi^2$$

then there exist two real negative and two complex roots, whose real parts are positive, according to the Hurwitz criterion. If

$$2\alpha^2 < \beta - \varepsilon_1 + \sqrt{(\beta - \varepsilon_1) + \tfrac{1}{2}\pi^2(\beta + \varepsilon_1) + \tfrac{1}{16}\pi^4} - \tfrac{1}{4}\pi^2$$

then all four roots are complex conjugate. The conditions were derived from the inequality $y_1(-a) < y_2(-a)$. The inverse Laplace transformation will yield the original function

$$q(t) = \frac{\psi(\lambda_1)}{\varphi'(\lambda_1)}e^{\lambda_1 t} + \frac{\psi(\lambda_2)}{\varphi'(\lambda_2)}e^{\lambda_2 t} + \frac{\psi(\lambda_3)}{\varphi'(\lambda_3)}e^{\lambda_3 t} + \frac{\psi(\lambda_4)}{\varphi'(\lambda_4)}e^{\lambda_4 t}$$

where λ_1, λ_2, λ_3, and λ_4 are the roots of function (4.10). The replacement of the kernel by (4.9) is equivalent to neglecting in the solution harmonics with a high frequency and small amplitude.

I am grateful to Kh. A. Rakhmatulin for the attention he has paid to my work.

BIBLIOGRAPHY

1. Rakhmatulin, Kh. A., Reshenie zadachi ob otrazhenii zvukovykh voln ot zhestkoi ploskosti, imeiushchei deformiruemuiu chast' (Solution of the problem of the reflection of sound waves from a rigid plane with a deformable portion). *PMM* Vol. 18, No. 4, 1954.

2. Miunts, *Integral'nye uravneniia (Integral Equations)*. GTTI, 1934.

Translated by M.I.Y.